ON A PROBLEM OF NACHBIN CONCERNING EXTENSION OF OPERATORS

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ABSTRACT

Problems I and II, stated below, are considered. It is shown that the answer to I may be negative even if X and Z are finite-dimensional and that the answer to II may be negative even if X and Z are separable and T compact. Concerning problem II some positive results are also obtained. For example, the answer to II is in the affirmative if X is a conjugate space or an L_1 space or if $X = c$ or c_0 and Z is separable.

1. Introduction. In the present note we are concerned with problem (6) of Nachbin [11]. We found it convenient to divide the problem into the following two parts.

I. *Let* Z, W *and* X *be Banach spaces with* $Z \supset W$ *and* $\dim Z/W = 2$. *Let* T *be an operator from* W *into* X. *Suppose that for every* Y *with* $Z \supset Y \supset W$ *and* $\dim Y/W = 1$ *there is a norm preserving extension of* T *from* Y *into* X. *Does* T *have a norm preserving extension from* Z *into* X?

II. *Let* Z, W *and* X *be Banach spaces with* $Z \supset W$ *and* $\dim Z/W = \infty$. *Let* T *be a bounded linear operator from* W *into* X. *Suppose that for every* Y *with* $Z \supset Y \supset W$ *and* $\dim Y/W < \infty$ *there is a norm preserving extension of* T *from* Y *into* X. *Does* T *have a norm preserving extension from* Z *into* X?

In Section 2 we make some simple observations which show that the answer to both questions is, in general, in the negative. We shall see, however that in many situations the answer to problem II is in the affirmative. In Section 3 we show that the answer to II may be negative even if the spaces Z and X are separable and T is compact. The construction used in this section is similar to that used in [8] for giving a counterexample to a question closely related to I.

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NOTATIONS. All operators are assumed to be linear and bounded. All Banach spaces are assumed to be over the reals (this is only a matter of convenience, all the results proved here hold also in the complex case). The unit cell $\{x; \|x\| \leqq 1\}$ of a Banach space X is denoted by S_X. Our notation for special spaces as $L_1, C(K)$,

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m and c_0 is standard. A Banach space X is called a $\mathfrak{P}_\lambda$ space if from every $Z \supset X$ there is a projection onto X of norm $\leqq \lambda$. For the basic facts concerning $\mathfrak{P}_\lambda$ spaces we refer to the book of Day [1, pp. 94–96]. A Banach space X is called an E_λ space if from every $Z \supset X$ with $\dim Z/X = 1$ there is a projection onto X of norm $\leqq \lambda$ (Grünbaum [4]). The projection constant $\mathfrak{P}(X)$ of X is defined by

$$\mathfrak{P}(X) = \inf\{\lambda;\ X \text{ is a } \mathfrak{P}_\lambda \text{ space}\}.$$

In a similar manner the expansion constant $E(X)$ of X is defined. The projection constant (or expansion constant) is said to be exact if the inf appearing in its definition is attained. We say that the Banach space X has the metric approximation property if for every compact set $K \subset X$ and every $\varepsilon > 0$ there is an operator T from X into itself with a finite-dimensional range satisfying $\| T \| = 1$ and $\| Tx - x \| \leqq \varepsilon$ for $x \in K$. This notion was introduced by Grothendieck [2]. It is not known whether there exists a Banach space which does not have this property. Let f be a functional defined on Y and let $X \subset Y$. The restriction of f to X is denoted by $f_{|X}$. Similarly we denote restrictions of operators.

2. We begin with some positive results

Proposition 1. *The answer to problem* II *is in the affirmative if X is a conjugate Banach space.*

Proof. This is a simple consequence of Tychonoff's theorem and the w^* compactness of the unit cell of X. The details of the proof are identical with those given in the proof of (4) ⇒ (9) in Theorem 2.2 of [5].

Corollary 1. *The answer to problem* II *is in the affirmative if there is a conjugate space $V \supset X$ and a projection P of norm* 1 *from V onto X.*

Proof. We extend T first in a norm preserving manner to an operator from Z into V and then apply P.

Remark. If X is an L_1 space then, as well known, there is a projection P with norm 1 from X^{**} onto X, and hence the answer to problem II is in the affirmative for such X. It is easy to see that in general if X is a Banach space and if there is a conjugate space $V \supset X$ from which there is a projection onto X with norm λ then there is also a projection with norm $\leqq \lambda$ from X^{**} onto X.

As we shall see in the next section the answer to problem II may be negative even for compact T. However, it follows easily from Proposition 1 that for compact T a slightly weaker version of II has an affirmative answer.

Corollary 2. *Let the assumptions on Z, W, X and T be as in problem* II. *Suppose further that T is compact and that X has the metric approximation property. Then for every $\varepsilon > 0$ there is an operator $\tilde{T}$ from Z into X with $\| \tilde{T} \| = \| T \|$ and $\| \tilde{T}_{|W} - T \| \leqq \varepsilon$.*

Proof. Since T is a compact and X has the metric approximation property there is an operator T_0 from X into itself having a finite-dimensional range B such that $\| T_0 \| = 1$ and $\| T_0T - T \| \leqq \varepsilon$. By Proposition 1 T_0T has an extension $\tilde{T}$ from Z into B with $\| \tilde{T} \| \leqq \| T \|$ and this proves the corollary(2).

We pass now to some counterexamples. We need two simple lemmas, the first of which is well known and therefore we omit its proof.

LEMMA 1. *Let $Z \supset W$ be Banach spaces with norm $\| \quad \|$ and unit cells S_Z and S_W respectively. Let $\lambda > 1$. We define in Z a new norm $|||\quad|||$ by taking as its unit cell the closed convex hull of $\lambda^{-1}S_Z \cup S_W$. Then*

(a) $\lambda |||z||| \geqq \lambda \| z \| \geqq |||z|||, \qquad z \in Z$

(b) $|||w||| = \| w \| \qquad , \qquad w \in W$

(c) *Let Y satisfy $Z \supset Y \supset W$ and let P be a projection from Y onto W. Then $|||P||| = 1$ if and only if $\| P \| \leqq \lambda$(3).*

LEMMA 2. *Let W be a Banach space and let $\lambda \geqq 1$. There exists a Banach space $X \supset W$ having the following property: Let Z be any Banach space containing W. There is a projection of norm $\leqq \lambda$ from Z onto W if and only if the identity operator from W into X has a norm preserving extension from Z into X.*

Proof. Let V be a $\mathfrak{P}_1$ space containing a subspace isometric to W, and let T_0 be an isometry from W into V. Let $X = V \oplus W$ where the norm is defined by $\| (v, w) \| = \max(\| v \|, \| w \| / \lambda)$. Let T_1 be the identity operator of W and let T from W into X be defined by $Tw = (T_0w, T_1w)$. T is an isometry. Let now Z be any space containing W. Since V is a $\mathfrak{P}_1$ space T_0 has a norm preserving extension from Z into V. T_1 has an extension with norm η from Z into W if and only if there is a projection of norm η from Z onto W. Therefore T has a norm preserving extension from Z into X if and only if there is a projection of norm $\leqq \lambda$ from Z onto W. This concludes the proof of the lemma (we identify W with the subspace TW of X).

Let W be a Banach space such that $\mathfrak{P}(W) > E(W)$. It may happen that from every $Z \supset W$ with $\dim Z/W = 2$ there is a projection with norm $\leqq E(W)$ onto W (take for example $W = c_0$). However this seems to be an exceptional case. We do not know of general results in this direction but it is easy to give examples. For instance it is not difficult to construct a 4-dimensional space Z containing the 2-dimensional inner product space W such that there is no projection from Z

(2) Actually, since $\|T_0T\|$ may be smaller than $\|T\|$, we apply here the following version of Proposition 1. Let $Z \supset W$ and X be Banach spaces with X being a conjugate space. Let T be an operator from W into X and let $\lambda \geqq \|T\|$ be given. If for every Y with $Z \supset Y \supset W$ and $\dim Y/W < \infty$ there is an extension of T from Y into X with norm $\leqq \lambda$, then there is also an extension of T from Z into X with norm $\leqq \lambda$. The proof of this assertion is the same as that of Proposition 1.

(3) $|||P||| = \text{sub} |||Py|||$ taken over all y with $|||y||| = 1$.

onto W with norm $\leqq E(W)$ $(= 2/\sqrt{3}$, cf. Grünbaum [3], [4]). Put $\lambda_0 = \inf\{\,\|P\|;$ P is a projection from Z onto $W\}$. By Lemmas 1 and 2 (with λ satisfying $E(W) \leqq \lambda < \lambda_0$) we obtain the following

(i) *There exist Banach spaces* $Z \supset W$ *with* $\dim Z = 4$, $\dim W = 2$ *such that there is no projection of norm* 1 *from* Z *onto* W *but for every 3-dimensional* Y *with* $Z \supset Y \supset W$ *there is a projection of norm* 1 *from* Y *onto* W.

(ii) *There exist finite dimensional Banach spaces* Z, W *and* X *with* $Z \supset W$, $X \supset W$, $\dim Z = 4$ *and* $\dim W = 2$ *such that the identity operator of* W *has a norm preserving extension from every 3-dimensional* Y, *containing* W, *into* X *but there is no such extension from* Z *into* X.

That in (ii) also X may be taken to be finite dimensional follows from the fact that the property of the 2-dimensional inner product space described above is shared also by 2-dimensional spaces sufficiently close to it. So we may take as W a 2-dimensional space whose unit cell is a regular polygon with sufficiently many sides. For such a W the space X constructed in Lemma 2 may be taken to be finite-dimensional.

These examples show that the answer to problem I is, in general, negative. By using the fact that there is no projection from m onto c_0 but that for every seperable $Y \supset c_0$ there is a projection of norm $\leqq 2$ from Y onto c_0 (Sobczyk [12]), we obtain from Lemmas 1 and 2 similar counterexamples to problem II.

It is interesting to compare these examples with the following, well known, result (this result motivated the question of Nachbin considered here). Let X be a Banach space. If $E(X) = 1$ and is exact then $\mathfrak{P}(X) = 1$ and is exact (this result holds also without the assumption of the exactness of $E(X)$, cf. [6]). We reformulate this result as follows. Let T be an isometry from W onto X. Let $Z \supset W$. If for every $Y \supset W$ with $\dim Y/W = 1$ T has a norm preserving extension from Y into X then T has also a norm preserving extension from Z into X.

The examples given above show that this is no longer true if we assume only that T is an isometry into or if we require only that T has a norm preserving extension from every Y with $Z \supset Y \supset W$ and $\dim Y/W = 1$ (or even from Y with $Z \supset Y \supset W$ and $\dim Y/W < \infty$).

Many results, showing that in certain special situations the answer to I or II or closely related questions is positive, are contained implicitly or follow easily from relations between certain extension properties and from characterization theorems proved in [6], [7], [9] and [10]. We give here only one example to illustrate this point.

Proposition 2. *Let* X *be a Banach space such that* $S_X{}^*$ *is* w^* *sequentially compact* (*in particular* X *may be any separable or reflexive space*) *and let* $W \subset X$. *If for every* $Y \supset W$ *with* $\dim Y/W < \infty$ *there is an operator with norm*

1 from Y into X whose restriction to W is the identity then the same is true for every $Y \supset W$ (without any restriction on Y/W).

Proof. By Theorem 3 of [9] it follows that the assumptions in Proposition 2 (even if we consider only $Y \supset W$ with $\dim Y/W = 1$) imply that W is finite-dimensional and its unit cell is a polyhedron. Hence there is a finite-dimensional $\mathfrak{P}_1$ space Y_0 containing W. Let T_0 be an operator with norm 1 from Y_0 into X whose restriction to W is the identity. Let Y be any space containing W. There is an operator T_1 with norm 1 from Y into Y_0 whose restriction to W is the identity. $T = T_0 T_1$ is the required operator from Y into X.

REMARKS. (a). Example (ii) above shows that Proposition 2 no longer holds if we require only that for every $Y \supset W$ with $\dim Y/W = 1$ there is an operator with norm 1 from Y to X whose restriction to W is the identity.

(b) Lemma 2 and the result of Sobczyk mentioned above show that we cannot discard the requirement on $S_X{}^*$ in the statement of Proposition 2.

3. The counterexamples to problem II given in the previous section were based on the theorem of Sobczyk and therefore the space Z had to be non separable. We shall now construct an example in which all the spaces are separable. We introduce first some notations. As mentioned in the introduction we use a construction similar to that in [8]. The notations will be the same as in [8] but the arguments used in the proofs and the purposes of the examples are quite different.

Let $\tilde{K}$ be the compact metric space of all the ordinals $\leqq \omega^2$ in the order topology(4). Let K_m, $m = 1, 2, \cdots$ be the subset of $\tilde{K}$ defined by

$$K_m = \{\alpha ; (m-1)\omega < \alpha \leqq m\omega\}. \tag{1}$$

Clearly $\tilde{K} - \{\omega^2\} = \cup_{m=1}^{\infty} K_m$(5). Let N denote the set of positive integers. Let $h(\alpha)$ be the function on $\tilde{K}$ defined by

$$h(\alpha) = \begin{cases} 1 \text{ if } \alpha = m\omega + 2j - 1, & m = 0,1,2,\cdots,\ j = 1,2,\cdots \\ -1 \text{ otherwise.} \end{cases} \tag{2}$$

Further let f_n, $n \in N$ be a sequence of continuous functions on $\tilde{K}$ defined by

$$f_n(\alpha) = \begin{cases} -1 & \text{if } \alpha \in K_{2m-1} \quad m = 1,2,\cdots,n \\ 1 & \text{otherwise.} \end{cases} \tag{3}$$

Let V be the space of all the bounded real-valued functions on (the abstract set) $\tilde{K} \times N$, with the usual vector operations and with the sup as norm. Let X_0 be the

(4) ω denotes, as usual, the ordinal number of the well-ordered set of the integers.

(5) $\{\omega^2\}$ denotes the set consisting of the single point ω^2. We do not consider here 0 as an ordinal number.

subspace of V consisting of all the functions v satisfying $v(\alpha, n) = v(\alpha, 1)$ for every $\alpha \in \tilde{K}$ and $n \in N$, and $v(\alpha, 1) \in C(\tilde{K})$. The mapping T_0 from X_0 onto $C(\tilde{K})$ defined by

(4) $$T_0 x(\alpha) = x(\alpha, 1) \qquad x \in X_0, \quad \alpha \in \tilde{K}$$

is clearly an isometry. Let Z_0 be the closed subspace of V spanned by X_0 and the functions

(5) $$z_0(\alpha, n) = f_n(\alpha) \qquad \alpha \in \tilde{K}, \quad n \in N$$

and

(6) $$z_k(\alpha, n) = \delta_{k,n} h(\alpha) \qquad \alpha \in \tilde{K}, \; n \in N, \; k = 1, 2, \cdots (^6)$$

With these notations we have the following

LEMMA 3. (a) *There is no projection from* Z_0 *onto* X_0 *with norm* $\leqq 5/4$,

(b) *For every* Y *with* $Z_0 \supset Y \supset X_0$ *and* $\dim Y/X_0 < \infty$, *and for every* $\varepsilon > 0$ *there is a projection of norm* $\leqq 1 + \varepsilon$ *from* Y *onto* X_0.

Proof. (a) Let P be a projection from Z_0 onto X_0 with $\| P \| = \lambda$. Let g_m, $m = 1, 2, \cdots$ be the characteristic function of the set $K_{2m} \times N$. $g_m \in X_0$ and $\| 2g_m - z_0 \| = 1$ for every m. Hence $\| 2g_m - Pz_0 \| \leqq \lambda$ and thus $Pz_0(\alpha, n) \geqq 2 - \lambda$ for $\alpha \in \bigcup_{m=1}^{\infty} K_{2m}$ and $n \in N$. By continuity of $Pz_0(\alpha, 1)$ we obtain

(7) $$Pz_0(\omega^2, 1) \geqq 2 - \lambda.$$

Let now $g_{m,j}$, $m, j = 1, 2, \cdots$ denote the characteristic function of the set $\{\overline{m-1}\, \omega + j\} \times N$. All the $g_{m,j}$ belong to X_0 and we have

$$\| z_0 + z_1 + z_2 + \cdots + z_m + 2g_{2m+1,2j} \| = 2 \qquad m, j = 1, 2, \cdots$$

Hence

$$(Pz_0 + z_1 + \cdots + z_m)(\alpha, n) \leqq -2 + 2\lambda \text{ for } \alpha = 2m\omega + 2j,$$

and by continuity

(8) $$P(z_0 + z_1 + \cdots + z_m)((2m + 2)\omega, 1) \leqq -2 + 2\lambda.$$

With the same $g_{m,j}$ as above we have also

$$\| z_1 + z_2 + \cdots + z_m - g_{k,2j+1} \| = 1, \qquad m, j, k = 1, 2, \cdots$$

As above, we obtain from this that

(9) $$P(z_1 + z_2 + \cdots + z_m)(k\omega, 1) \geqq 1 - \lambda \qquad k, m = 1, 2, \cdots$$

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(6) $\delta_{n,k} = 1$ if $n = k$ and 0 otherwise.

By (8) and (9) $Pz_0((2m+1)\omega, 1) \leqq -3 + 3\lambda$ for every m and hence $Pz_0(\omega^2, 1) \leqq -3 + 3\lambda$. This together with (7) implies that $\lambda \geqq 5/4$.

We turn to the proof of (b). Assume first that Y is the span of X_0 and $\{z_i\}_{i=0}^k$ for some finite k. The operator $T_k z(\alpha) = z(\alpha, k+1)$ maps Y into $C(\tilde{K})$ and is clearly a norm preserving extension of T_0. Hence there is a projection of norm 1 from Y onto X_0. Let now Y be a general subspace of Z_0 containing X_0 as a subspace of finite deficiency. That is $Y = \mathrm{sp}\{X_0, b_1, \cdots b_k\}$ with $k = \dim Y/X_0$ and $b_i \in Z_0$. There is an $M < \infty$ such that

$$\sum_{i=1}^{k} |\lambda_i| \leqq M \left\| x + \sum_{i=1}^{k} \lambda_i b_i \right\|, \qquad x \in X_0, \lambda_i \text{ real}.$$

Let $\tilde{b}_i$, $i = 1, \cdots, k$ be in the dense subspace of Z_0 spanned (linearly not topologically) by X and $\{z_j\}_{j=0}^{\infty}$, such that $\|\tilde{b}_i - b_i\| < \varepsilon/M$. By what we have already shown there is a projection $\tilde{P}$ of norm 1 from the subspace of Z_0 spanned by X_0 and $\{\tilde{b}_i\}_{i=1}^k$ onto X_0. Define now P from Y onto X_0 by

$$P(x + \Sigma_i \lambda_i b_i) = \tilde{P}(x + \Sigma_i \lambda_i \tilde{b}_i).$$

We have(7)

$$\|P(x + \Sigma_i\lambda_i b_i)\| \leqq \|x + \Sigma_i \lambda_i \tilde{b}_i\| \leqq$$

$$\leqq \|x + \Sigma_i \lambda_i b_i\| + \varepsilon \Sigma_i |\lambda_i|/M \leqq (1+\varepsilon)\|x + \Sigma_i \lambda_i b_i\|.$$

Hence P is a projection of norm $\leqq 1 + \varepsilon$.

REMARKS. 1. The question whether and when we can take $\varepsilon = 0$ in (b) was treated in [8].

2. It is clear that a construction similar to that done in Lemma 3 can be done for every compact metric K with $(K')' \neq \emptyset$(8). If we use for these K exactly the same construction (with the obvious modification obtained by replacing the characteristic functions of the sets K_m by suitable Urysohn functions) we will get of course the same constant (that is 5/4) in (a). It seems likely that if we consider the spaces of ordinals $\leqq \omega^k$ it is possible to construct similar examples with 5/4 replaced by a number γ_k tending to ∞ with k, and thus by taking direct sums we would get an example in which 5/4 can be replaced by ∞ (i.e. in which there is no bounded projection at all from Z_0 into X_0). We have, however, not worked out the details of such constructions.

Combining Lemma 3 (cf. also remark 2) with Lemma 1 we get

(7) This inequality shows also that P is well defined, i.e. that no non trivial combination of the $\tilde{b}_i$ belongs to X_0.

(8) K' denotes the set of limiting points of K.

PROPOSITION 3. *There is a $\lambda > 1$ such that for every compact metric K with $(K')' \neq \emptyset$ there exists a separable $Z \supset C(K)$ satisfying*

(a) *There is no projection of norm $\leqq \lambda$ from Z onto $C(K)$.*

(b) *From every Y with $Z \supset Y \supset C(K)$ and $\dim Y/C(K) < \infty$ there is a projection of norm 1 from Y onto $C(K)$.*

Clearly, every $\lambda < 5/4$ will do. Proposition 3 does not hold if $(K')' = \emptyset$. Indeed, we have

PROPOSITION 4. *In problem* II *let $X = C(K)$ with K compact metric. The answer to the problem is in the affirmative for every separable Z and W and for every T if and only if $(K')' = \emptyset$.*

Proof. That the answer to II may be negative even for separable Z if $(K')' \neq \emptyset$ follows from Proposition 3. That the answer is always in the affirmative if $(K')' = \emptyset$ is is an easy consequence of

LEMMA 4. *Let $Z \supset W$ be separable Banach spaces and let $\{f_n\}_{n=1}^{\infty}$ be a w^* convergent sequence in $S_W{}^*$. Suppose that for every Y with $Z \supset Y \supset W$ and* $\dim Y/W < \infty$ *there is a w^* convergent sequence $\{y_n^*\}_{n=1}^{\infty}$ in $S_Y{}^*$ such that $y^*_{n|W} = f_n$ for every n. Then there is a w^* convergent sequence $\{z_n^*\}_{n=1}^{\infty}$ in $S_Z{}^*$ such that $z^*_{n|W} = f_n$ for every n.*

Proof. Let $Z = \mathrm{sp}(W, \{z_i\}_{i=1}^{\infty})$ and put $Y_m = \mathrm{sp}(W, \{z_i\}_{i=1}^{m})$, $m = 1, 2, \cdots$. Let $\{y^*_{m,n}\}_{n=1}^{\infty} \in S_{Y_m^*}$ be a w^* convergent sequence (to y^*_m, say) such that $y^*_{m,n\,|\,W} = f_n$. By the diagonal method we choose a sequence m_j such that $\lim_{j\to\infty} y^*_{m_j}(z_i)$ exists for every i([9]), and call this limit $z^*(z_i)$. Next we choose an increasing sequence $\{n_k\}_{k=1}^{\infty}$ of integers and a subsequence $\{m_k\}_{k=1}^{\infty}$ of $\{m_j\}_{j=1}^{\infty}$ (for simplicity of notation we do not add another index) such that

$$|\, y^*_{m_k}(z_i) - z^*(z_i) \,| < k^{-1} \qquad \text{for } i < k$$

$$|\, y^*_{m_k,n}(z_i) - y^*_{m_k}(z_i) \,| < k^{-1} \qquad \text{for } n > n_k, \quad i < k.$$

Let now z_n^* be any norm preserving extension of $y^*_{m_k,n}$ to Z if $n_k < n \leqq n_{k+1}$, $k = 1, \cdots$. It is easily verified that the sequence $\{z_n^*\}_{n=1}^{\infty}$ has the required properties.

We conclude this paper by showing that the answer to problem II may be negative even for operators with a finite-dimensional range (and hence, in particular, for compact operators).

PROPOSITION 5. *There exist separable Banach spaces Z, W and X with $Z \supset W$, $X \supset W$ and* $\dim W = 2$ *such that*

([9]) $y^*_{m_j}(z_i)$ is defined if $m_j \geqq i$.

(a) *There is no operator with norm* 1 *from* Z *into* X *whose restriction to* W *is the identity.*

(b) *For every finite-dimensional* Y *with* $Z \supset Y \supset W$ *there is an operator with norm* 1 *from* Y *into* X *whose restriction to* W *is the identity.*

Proof. Let $X = C[0,\pi/2]$ and let $Z \supset X$ be the space constructed in Proposition 3. Let W be the subspace of X spanned by $w_1(t) = \cos t$ and $w_2(t) = \sin t$. We prove (b) first. Let Y be finite-dimensional with $Z \supset Y \supset W$. Let Y_0 be the subspace of Z spanned by X and Y. By Proposition 3 (b) there is a projection of norm 1 from Y_0 onto X. The restriction of this projection to Y has the required properties.

Proof of (a). For $t \in [0,\pi/2]$ let $\phi_t \in X^*$ be defined by $\phi_t(x) = x(t)$, $x \in X$. For every t there is a $w_t \in W$ such that $w_t(t) = 1$ and $|w_t(s)| < 1$ for $s \neq t$. It follows that ϕ_t is the unique norm preserving extension of $\phi_{t\,|\,W}$ into X. Suppose there were an operator T with norm 1 from Z into X whose restriction to W is the identity. Let $z_t^* = T^*\phi_t$, $t \in [0,\pi/2]$. We have $\| z_t^* \| \leqq 1$ and $z^*_{t\,|\,W} = (T^*\phi_t)_{|W} = \phi_{t\,|\,W}$ (since $T_{|\,W}$ is the identity). It follows that $z^*_{t\,|\,X}$ is a norm preserving extension of $\phi_{t\,|\,W}$ and hence, as observed above, $z^*_{t\,|\,X} = \phi_t$. Let now $x \in X$. Then

$$Tx(t) = \phi_t(Tx) = T^*_{\phi_t}(x) = z_t^*(x) = z^*_{t\,|\,X}(x) = \phi_t(x) = x(t).$$

Thus $Tx = x$, in other words T is a projection and this contradicts Proposition 3 (a).

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